or

$$\frac{-1}{4\sin A\sin B\sin C}\begin{vmatrix} \sin^2 A+\sin^2 B+\sin^2 C, & \sin^2 A+\sin^2 B+\sin^2 C, & \sin^2 A+\sin^2 B+\sin^2 C \\ 1, & 1, & 1 \\ 2\sin^2 A, & 2\sin^2 B, & 2\sin^2 C \end{vmatrix}.$$

But this is equal to zero since two rows are alike after dividing out $\sin^2 A + \sin^2 B + \sin^2 C$. It is to be noticed also that the above determinant is equal to zero for any values whatever of A, B, C provided only that two of them are alike.

Also solved by ELIJAH SWIFT, G. W. HARTWELL, H. POLISH, R. M. MATHEWS, H. L. AGARD, H. S. UHLER, CLIFFORD N. MILLS, W. W. BURTON, CARL A. W. STROM, A. M. KENYON, J. H. WEAVER, and A. H. WILSON.

CALCULUS.

387. Proposed by C. N. SCHMALL, New York City.

Show that the volume bounded by the cone $x^2 + y^2 = (a - z)^2$ and the planes $x = 0$, $x = z$ is $\frac{2}{9}a^3$.

I. SOLUTION BY A. M. HARDING, University of Arkansas.

The projection on the XY-plane of the curve of intersection of the cone and the plane $z = x$ is $y^2 = a^2 - 2ax$.

If we change this equation to polar coördinates we obtain

$$\rho = \frac{a}{1+\cos\theta} = \frac{a}{2}\sec^2\frac{\theta}{2}.$$

Hence,

$$\frac{v}{2} = \int_0^{\pi/2}\int_0^{a/2\sec^2\theta/2}\int_x^{a-\sqrt{x^2+y^2}} dz \cdot \rho d\rho d\theta = \int_0^{\pi/2}\int_0^{a/2\sec^2\theta/2}(a - \sqrt{x^2+y^2} - x)\rho d\rho d\theta$$

$$= \int_0^{\pi/2}\int_0^{a/2\sec^2\theta/2}(a\rho - \rho^2 - \rho^2\cos\theta)d\rho d\theta = \frac{a^3}{24}\int_0^{\pi/2}\sec^4\frac{\theta}{2}d\theta = \frac{a^3}{9}.$$

Hence the entire volume is $v = \frac{2}{9}a^3$.

II. SOLUTION BY GEO. W. HARTWELL, Hamline University.

If this volume is sliced parallel to the xy plane, the sections between $z = 0$ and $z = a/2$ will be segments of circles and from $z = a/2$ to $z = a$ semicircles.

Integrating then we have

$$\int_0^{a/2}\left[\tfrac{1}{2}\pi(a-z)^2 + z\sqrt{a^2-2az} - (a-z)^2\cos^{-1}\frac{z}{a-z}\right]dz + \int_{a/2}^{a}\tfrac{1}{2}\pi(a-z)^2 dz$$

$$= \int_0^a \tfrac{1}{2}\pi(a-z)^2 dz + \int_0^{a/2}\left[z\sqrt{a^2-2az} - (a-z)^2\cos^{-1}\frac{z}{a-z}\right]dz.$$

Integrating we have

$$[-\tfrac{1}{6}\pi(a-z)^3]_{z=0}^{z=a} + \Bigg[-\frac{(2a^2+6az)(a^2-2az)^{3/2}}{30a^2} + \frac{(a-z)^3}{3}\cos^{-1}\frac{z}{a-z} - \frac{a^2\sqrt{a^2-2az}}{3}$$

$$+ \frac{(2a^2+2az)\sqrt{a^2-2az}}{9} - \frac{(2a^2+2az+3z^2)\sqrt{a^2-2az}}{45}\Bigg]_{z=0}^{z=(a/2)} = \frac{2}{9}a^3.$$

Also solved by H. L. AGARD, FRANK R. MORRIS, NORMAN ANNING, H. S. UHLER, and the PROPOSER.

388. Proposed by PAUL CAPRON, U. S. Naval Academy.

If $f(x, y) = 0$ represents a curve having a simple tangency to the axis of x at the origin, the

value of $x^2/2y$, derived from $f(x, y) = 0$, and evaluated for $x = 0$, $y = 0$, will be the radius of curvature at the origin; or if the curve is similarly tangent to the y-axis at the origin, $y^2/2x$, evaluated for $x = 0$, $y = 0$, is the radius of curvature at the origin.

Solution by A. M. Harding, University of Arkansas.

The equation of any curve having simple tangency to the axis of x at the origin may be written in the form

$$f(x, y) = Ay + \tfrac{1}{2}Bx^2 + Cxy + \tfrac{1}{2}Dy^2 + \cdots = 0,$$

where $A, B, C, D, \cdots$ are constants and $A \neq 0$, $B \neq 0$.

The radius of curvature at any point is given by

$$r^2 = \frac{[f_x^2 + f_y^2]^3}{[f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2]^2}, \qquad \text{where} \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \text{ etc.}$$

When $x = 0$ and $y = 0$, we find

$$f_x = 0, \quad f_y = A, \quad f_{xx} = B, \quad f_{xy} = C, \quad f_{yy} = D.$$

Substituting, we obtain $r^2 = A^2/B^2$ or $r = A/B$. Dividing the given equation by B, we have

$$\frac{A}{B} + \frac{x^2}{2y} + Cx + \frac{Dy}{2B} + \cdots = 0.$$

Letting x and y approach zero,

$$\frac{A}{B} = -\left[\frac{x^2}{2y}\right]_{\substack{x=0\\y=0}}.$$

Hence,

$$r = -\left[\frac{x^2}{2y}\right]_{\substack{x=0\\y=0}}.$$

In the second case the method is the same and the equation has the form

$$f(x, y) = Ax + \tfrac{1}{2}Bx^2 + Cxy + \tfrac{1}{2}Dy^2 + \cdots = 0.$$

Also solved by Horace Olson in a special form and with incorrect interpretation of "simple tangency."

389. Proposed by FRANK R. MORRIS, Glendale, Calif.

A man is at the southeast corner of a section of land and wishes to walk to the opposite corner in the least possible time. A circular track with a radius of $1/\pi$ miles is located in the section tangent to the west line at a point 120 rods from the south line. Conditions are such that he can walk at the rate of 4 miles an hour inside the track and 3 miles an hour outside the track. What course should he choose and how long is it?

I. Solution by H. S. Uhler, Yale University.

Since the problem involves rectilinear motion at different speeds it is a question of refraction and can be solved at once by the methods of geometrical optics; for, by Fermat's principle, the time taken by light in going from the southeast to the northwest corner of the section will be either a minimum or a maximum (in this case, a minimum). The index of refraction of the medium outside the circle relative to the medium inside the circle is 4/3 (water and air, say). Hence, the "optical invariant" is

$$3 \sin i = 4 \sin r. \tag{1}$$

By hypothesis, (Fig. 1).

$\overline{OW} = \overline{WN} = 1$ mile. $\overline{CA} = \overline{CB} = \overline{CT} = 1/\pi$ miles, and $\overline{PC} = 120$ rods $= \frac{3}{8}$ mile.

The $\triangle ACB$ is isosceles so that $\angle CBA = \angle CAB \equiv i$. Hence, by equation (1) $\angle EBN = \angle DAO \equiv r$.

The projection of the broken line $OPCA$ on a diameter $\overline{FG}$ perpendicular to $\overline{OA}$ equals zero.